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On Minimum Phase

Über Minimalphasigkeit

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Wir diskutieren Minimalphasigkeit von schwach-stabilen Transferfunktionen; letzteres sind rationale Funktionen, bei denen das Nennerpolynom Nullstellen in der abgeschlossenen linken komplexen Halbebene hat. Minimalphasigkeit wird hier mittels der Ableitung der Argumentfunktion der Transferfunktion definiert. Es wird dann mit Hilfe der Hurwitz-Reflektion gezeigt, daß jede schwach-stabile Transferfunktion eindeutig in ein Produkt von Allpass und minimalphasiger Funktion zerlegt werden kann. Das wesentliche Resultat ist, daß eine schwach-stabile Transferfunktion minimalphasig ist genau dann, wenn das Zählerpolynom der Transferfunktion schwach-stabil ist. Ein weiteres Resultat ist, daß die Nulldynamik einer minimalen Realisation asymptotisch stabil ist genau dann, wenn das Zählerpolynom der Transferfunktion Hurwitz ist. Insbesondere folgt aus asymptotisch stabiler Nulldynamik die Minimalphasigkeit, aber keineswegs umgekehrt. Abschließend zeigen wir, daß ein minimalphasiges System als kanonischer Repräsentant innerhalb der Äquivalenzklasse aller Systeme mit identischem Betragsverhalten interpretiert werden kann.

We discuss the concept of 'minimum phase' for weakly stable transfer functions. The latter are rational functions where the denominator polynomial has its roots in the closed left half complex plane. In the present note, minimum phase is defined in terms of the derivative of the argument function of the transfer function. The main tool to characterize minimum phase is the Hurwitz reflection. The factorization of a weakly stable transfer function into an all-pass and a minimum phase system leads to the result that any weakly stable transfer function is minimum phase if, and only if, its numerator polynomial is weakly Hurwitz. We show in particular that asymptotic stable zero dynamics of a minimal realization of a transfer function yields minimum phase, but not vice versa. Finally, we interpret a minimum phase system as a canonical representative within the class of all gain equivalent systems.

Schlagwörter: Minimalphasigkeit, Allpassfilter, Hurwitz-Spiegelung, inner-outer-Faktorisierung, Nulldynamik, kanonische Form.

Keywords: Minimum phase, all-pass, Hurwitz reflection, inner-outer factorization, zero dynamics, canonical form.

Nomenclature

$\mathbb{K}$	is either the field of real numbers $\mathbb{R}$ or of complex numbers $\mathbb{C}$
$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{C}_+ (\overline{\mathbb{C}}_+)$	open (closed) set of complex numbers with positive (non-negative) real part, resp.
$\mathbb{C}_- (\overline{\mathbb{C}}_-)$	open (closed) set of complex numbers with negative (non-positive) real part, resp.
$\imath$	imaginary unit in $\mathbb{C}$
$\mathbf{Gl}_n(\mathbb{K})$	the group of invertible $n \times n$ matrices with entries in $\mathbb{K}$
$\mathbb{K}[s]$	the ring of polynomials with coefficients in $\mathbb{K}$
$\mathbb{K}[s]_{\text{monic}} :=$	$\{p(s) \in \mathbb{K}[s] \mid p(s)$ has leading coef. $1\}$, the set of monic polynomials
$\mathbb{K}(s)$	the quotient field of $\mathbb{K}[s]$, i.e. the set of rational functions
$\mathbb{K}(s)_{\text{pr.}} :=$	$\{\frac{p(s)}{q(s)} \in \mathbb{K}(s) \mid \deg q(s) \geq \deg p(s)\}$, the set of *proper* rational functions
$\mathbb{K}(s)_{\text{str.pr.}} :=$	$\{\frac{p(s)}{q(s)} \in \mathbb{K}(s) \mid \deg q(s) > \deg p(s)\}$, the set of *strictly proper* rational functions
$R^{n \times m}$	the set of $n \times m$ matrices with entries in a ring R
R^*	$R \backslash \{0\}$ for a ring R

Furthermore, a polynomial $p(s) \in \mathbb{C}[s]$ is called

$$\begin{aligned} \textit{Hurwitz} \quad &:\Longleftrightarrow \quad p(s_0) = 0 \ \Rightarrow \ \operatorname{Re} s_0 < 0 \\ \textit{anti Hurwitz} \quad &:\Longleftrightarrow \quad p(s_0) = 0 \ \Rightarrow \ \operatorname{Re} s_0 > 0 \\ \textit{weakly Hurwitz} \quad &:\Longleftrightarrow \quad p(s_0) = 0 \ \Rightarrow \ \operatorname{Re} s_0 \leq 0 \end{aligned}$$

and $g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)$ is called *weakly stable* if, and only if, the denominator polynomial $q(s)$ is weakly Hurwitz. We stress that the notion 'weakly' allows zeros on the imaginary axis, and hence a realization may be unstable.

The *relative degree of* $\frac{p(s)}{q(s)} \in \mathbb{C}(s) \backslash \{0\}$ is the number $\rho = \deg q(s) - \deg p(s) \in \mathbb{Z}$.

1 Introduction

We consider linear single-input single-output systems of the form

$$y(s) \;=\; g(s)\; u(s) \tag{1.1}$$

with weakly stable transfer function $g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s) \backslash \{0\}$.

The notion of minimum phase systems was introduced for the first time by Bode in [1] with the terse words "If the circuit includes no surplus lines or all-pass sections, it will have at every frequency the least phase shift (algebraically) which can be obtained from any physical structure having the given attenuation characteristic." This leads to the famous Bode gain-phase relation by which the phase is uniquely determined by the gain (i.e. the attenuation characteristic), provided that the phase represents the "minimum phase shift" that must be associated with the given gain.

In less quaint words: within the set of transfer functions $g(s) \in \mathbb{C}(s)$ of equal gain $|g(\imath\omega)|$ along the imaginary axis, we wish to single out the one transfer function minimizing the change of the argument $\arg g(\imath\omega_2) - \arg g(\imath\omega_1)$ for all $\omega_2 > \omega_1$.

This property is discussed briefly in most classical textbooks on control systems and today one frequently finds as the defining property that the zeros of numerator polynomial should lie in $\mathbb{C}_-$. The latter property is of interest because it guarantees stability of the zero dynamics. This has led to the abuse of terminology that a nonlinear systems is called "minimum phase" if the zero dynamics is asymptotically stable.

There seems to have been a development that separated the meaning of the term "minimum phase shift" from its everyday usage. The standard notion (zeros of the numerator polynomial in $\mathbb{C}_-$) is not directly linked to the phase property. This we do not consider as a definition of minimum phase since the definiendum (minimum phase) and definiens (some location of the zeros of a polynomial) do not have a term in common. One might justify such a definition, if at least it were possible to prove the equivalence of the two statements. However this is only the case, if we assume from the outset, that the transfer function does not have any zeros on the imaginary axis, as we will see below.

We thus advocate a distinction in terms: minimum phase systems can be characterized in terms of the phase - the desired property of stability of the zero dynamics, however, is not an equivalent property. Rather, there are two distinct properties and if stability of the zero dynamics is needed then this is what should be expressed.

We note in passing that another frequent slight of tongue is to associate properties of the step response with minimum phase. For instance, we may frequently read that initial undershoot in the step response is a "non-minimum phase characteristic". However, the results in [5] show that initial undershoot is not characterized equivalently by stability of the zero dynamics or for that matter by minimum phase.

We aim to define "minimum phase" in such a form that zeros of the numerator polynomial $p(s)$ in (1.1) are encompassed and the phase (where defined) of a minimum phase system is minimal under all gain equivalent transfer functions. The latter is stated precisely as follows.

Definition 1.1 *Two transfer functions $g_1(s), g_2(s) \in \mathbb{C}(s)$ are called* gain equivalent *(written $g_1(s) \overset{\text{ge}}{\sim} g_2(s)$) if, and only if,*

$$|g_1(\imath\omega)| = |g_2(\imath\omega)| \qquad \textit{for almost all } \omega > 0\,. \tag{1.2}$$

We note that the definition does not depend on properness or on the condition that there are no poles on $\imath\mathbb{R}$; indeed, the requirement that the equality only holds almost everywhere accounts for poles on $\imath\mathbb{R}$. If such poles do not exist, then continuity immediately implies that the equation holds everywhere.

Note that $\overset{\text{ge}}{\approx}$ defines an equivalence relation on $\mathbb{C}(s) \times \mathbb{C}(s)$ and we denote the *equivalence classes* by

$$[g(s)]_{\text{ge}} \;:=\; \{\tilde{g}(s) \in \mathbb{C}(s) |\; \tilde{g}(s) \overset{\text{ge}}{\approx} g(s)\}\,, \quad g(s) \in \mathbb{C}(s)\,.$$

The notion of minimum phase relies crucially on the concept of the phase or in mathematical terms of the argument of a transfer function. We recall (see e.g. [4, Proposition A.2.3]) that for any transfer function $\frac{p(s)}{q(s)} \in \mathbb{C}(s)\setminus\{0\}$ with $q(s)$ weakly Hurwitz there exists a differentiable function

$$\arg g(\imath\cdot) \;:\; \mathbb{R}\setminus\{\omega \in \mathbb{R}\,|p(\imath\omega)\,q(\imath\omega) = 0\,\} \to \mathbb{R}$$

such that

$$g(\imath\omega) = |g(\imath\omega)|\; \mathrm{e}^{\imath \arg g(\imath\omega)} \quad \forall\,\omega \in \mathbb{R} \;:\; p(\imath\omega)\,q(\imath\omega) \neq 0.$$

The derivative of $g(\imath\cdot)$ is unique since every argument function is unique up to a constant $2k\pi$, $k \in \mathbb{Z}$. In the following we will always consider argument functions that are differentiable on any open interval contained in the domain of definition.

The following property of the argument of polynomials is well known, see [4, Prop. 3.4.3]: Given a polynomial $p(s)$ without zeros on the imaginary axis, we have that the total change of the argument along the imaginary axis satisfies

$$\begin{aligned}\Delta_{-\infty}^{\infty}\; p(\imath\omega) &:= \lim_{\omega\to\infty}\big(\arg p(\imath\omega) - \arg p(-\imath\omega)\big)\\ &= (\deg p(s) - 2\nu)\pi\,,\end{aligned}$$

where ν is the number of zeros of $p(s)$ in $\mathbb{C}_+$ (counting multiplicities). For a proper stable transfer function $g(s) = p(s)/q(s)$ we then obtain immediately (using $\arg 1/q(s) = -\arg q(s)$) that

$$\Delta_{-\infty}^{\infty} g(\imath\omega) = (\deg p(s) - 2\nu_p - \deg q(s))\pi < 0\,,$$

where now ν_p denotes the number of zeros of p in $\mathbb{C}_+$. It is now obvious that the absolute value of the total change of the argument of $g(s)$ along the imaginary axis is minimal if $\nu_p = 0$.

This approach to a definition of minimum phase is unsatisfactory, as it excludes the possibility of zeros on the imaginary axis (in which case the necessary quantities are not well-defined). Also it does not capture the full strength of the minimum phase property, as the intermediate behaviour is not really taken into account. We learn however that the total phase change is a negative quantity and thus to keep the absolute value of this quantity small, we should aim at a derivative, which is as large as possible. This allows to suggest the following definition of minimum phase.

Definition 1.2 *A weakly stable transfer function $\widehat{g}(s) \in \mathbb{C}(s)\setminus\{0\}$ is said to be* minimum phase *if, and only if, all other gain equivalent weakly stable transfer functions $g(s) \in \mathbb{C}(s)$ satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\arg g(\imath\omega) \leq \frac{\mathrm{d}}{\mathrm{d}\omega}\arg\widehat{g}(\imath\omega) \quad \textit{for almost all}\,\omega \in \mathbb{R}.$$

Definition 1.2 says that within the equivalence class $[\widehat{g}(s)]_{\text{ge}}$ of all gain equivalent transfer functions with weakly Hurwitz denominator function, the transfer function $\widehat{g}(s)$ is minimum phase if, and only if, the change of its argument function is at almost each $\omega \in \mathbb{R}$ smaller or equal than the change of the argument function of any other function belonging to the same equivalence class. Therefore, for an appropriate choice of the argument functions, the phase of the minimum phase function is smaller than the phase of the non-minimum phase function. The typical picture of a minimum phase transfer function is depicted in Figure 1.1.1.

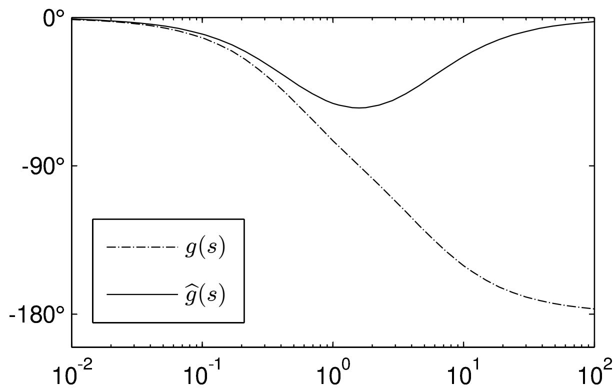

Bild 1.1.1: Minimum phase transfer function $\widehat{g}(s)$ and non-minimum phase transfer functions $g(s)$

Another example of minimum phase transfer functions is

$$g(s) = s - \imath\omega_0 \qquad \text{for some}\;\; \omega_0 \in \mathbb{R}\,.$$

Then, for arbitrary $k, \ell \in \mathbb{Z}$,

$$g(\imath\omega) \;=\; \begin{cases} (\omega - \omega_0)\; \mathrm{e}^{\imath\left(\frac{\pi}{2}+2k\pi\right)} & , \omega > \omega_0\\ 0 & , \omega = \omega_0\\ (\omega - \omega_0)\; \mathrm{e}^{\imath\left(\frac{3\pi}{2}+2\ell\pi\right)} & , \omega < \omega_0 \end{cases} \tag{1.3}$$

and the argument of $\omega \mapsto g(\imath\omega)$ is constantly $\pi/2$ on (ω_0, ∞), $3\pi/2$ on $(-\infty, \omega_0)$, and has a discontinuity at $\omega = \omega_0$.

In Figure 1.1.1 we have plotted the phase of two simple but prototypical examples:
$g(s) = (-0.2s + 1)/(2s + 1)$ is a non-minimum phase transfer function which is gain equivalent to $\widehat{g}(s) = (0.2s+1)/(2s+1)$, a minimum phase transfer function. Please note that the derivative of the argument of $g(\imath\cdot)$ is strictly smaller than the derivative of the argument

of $\widehat{g}(\imath\cdot)$. This confirms Definition 1.2 together with the result in Theorem 2.7.

The paper is organized as follows: In Section 2 we factorize, in terms of the Hurwitz reflection and all-passes, all weakly stable transfer functions. The main result is to show that a transfer functions is minimum phase if, and only if, its numerator polynomial is weakly Hurwitz. As a side result we show that a proper and stable rational function is inner (outer) if, and only if, it is an all-pass (minimum phase), resp. In Section 3 we show that asymptotic stable zero dynamics of a minimal realization of the transfer function yields minimum phase, but not vice versa. Finally, in Section 4 we show, loosely speaking, that a minimum phase system may be interpreted as the canonical form of an equivalence class of gain equivalent transfer functions.

What is new in the present contribution? Many of our observations are frequently encountered in textbooks in a more or less precise form; quite often phrases as 'it is easy to show' are used or the reader is left alone in another way. Whether or not zeros of the numerator polynomial of a stable transfer function are allowed for minimum phase systems is not consistently handled. The use of the derivative as in Definition 1.2 is, to the best of our knowledge, new. We are only aware of a paragraph in Unbehauen [7, p. 483] where it is mentioned, in passing, that a minimum phase system may be defined invoking the derivative.

As a consequence of the concentration of phase properties in the definition of minimum phase, it follows that stability of the zero dynamics is equivalent to the fact that the numerator polynomial is Hurwitz, which implies the minimum phase property but is not implied by it.

2 Minimum phase

The main result of this section is the characterization of the minimum phase property of weakly stable transfer functions $\frac{p(s)}{q(s)} \in \mathbb{C}(s)$ in Section 2.3. Before that, we collect basic properties of all-passes in Section 2.1 and show certain factorizations of weakly stable transfer functions in Section 2.2.

2.1 All-pass and Hurwitz reflection

In this section we characterize an all-pass in terms of Hurwitz-reflections.

Definition 2.1 *A transfer function $g(s) \in \mathbb{C}(s)$ is called an* all-pass *if, and only if,*

$$|g(\imath\omega)| = 1 \qquad \text{for almost all } \omega \in \mathbb{R}\,. \quad (2.1)$$

Again, the notion "almost all" mean finitely many as only the poles of $g(s)$ are affected by this.

A useful technical tool to investigate all-passes is the following mapping of polynomials.

Definition 2.2 *[4, Definition 3.4.4]*
The mapping

$$\begin{aligned} {}^\star \,:\, \mathbb{C}[s] &\to \mathbb{C}[s], \qquad (2.2)\\ \sum_{k=0}^n p_k s^k = p(s) &\mapsto p^\star(s) := \overline{p}(-s) = \sum_{k=0}^n (-1)^k \overline{p}_k s^k\,, \end{aligned}$$

is called Hurwitz reflection*.*

The Hurwitz reflection is an involutive $\mathbb{R}$-linear degree preserving bijection and satisfies, for all $\alpha \in \mathbb{C}$ and all $p(s), q(s) \in \mathbb{C}[s]$,

$$(\alpha\, p(s))^\star = \overline{\alpha}\, p^\star(s) \quad \text{and} \quad (p(s)q(s))^\star = p^\star(s)q^\star(s)$$

and

$$p^\star(s) = (-1)^n \overline{\gamma}_p \prod_{k=1}^n (s+\bar{s}_k)$$
$$\text{for} \qquad p(s) = \gamma_p \prod_{k=1}^n (s-s_k) \in \mathbb{C}[s]. \quad (2.3)$$

Therefore, $s_0 \in \mathbb{C}$ is a zero of $p(s)$ if, and only if, $-\overline{s}_0$, i.e. the reflection of s_0 at the imaginary axis, is a zero of $p^\star(s)$. If $p(s) \equiv p_0 \in \mathbb{C}$, then $p^\star(s) \equiv \bar{p}_0$. The mapping * maps any Hurwitz polynomial $p(s)$ to the *anti-Hurwitz polynomial* $p^\star(s)$, i.e. all zeros of $p^\star(s)$ lie in the open right half complex plane $\mathbb{C}_+$.

Proposition 2.3 *Any $r(s) \in \mathbb{C}[s]$ satisfies*

$$|r(\imath\omega)| \;=\; |r^\star(\imath\omega)| \qquad \forall\, \omega \in \mathbb{R}\,.$$

Proof We conclude, for $r(s)$ factorized as in (2.3), that

$$\begin{aligned} |r(\imath\omega)| &= \Big|\gamma \prod_{k=1}^n (\imath\omega - s_k)\Big| = \Big|\bar{\gamma} \prod_{k=1}^n \overline{(\imath\omega - s_k)}\Big| \\ &= \Big|\bar{\gamma} \prod_{k=1}^n (\imath\omega + \overline{s}_k)\Big| \;=\; |r^\star(\imath\omega)|\,. \quad \square \end{aligned}$$

We will now show that every all-pass has a simple representation.

Proposition 2.4 *For any $g(s) \in \mathbb{C}(s)$ we have*

$$g(s) \overset{\text{ge}}{\approx} 1 \iff g(s) \;=\; \mathrm{e}^{\imath\varphi}\, \frac{p(s)}{p^\star(s)}$$
for unique monic $p(s) \in \mathbb{C}[s]$ such that $p(\imath\mathbb{R}) \cap \{0\} = \emptyset$ and $\varphi \in [0, 2\pi)$.

Proof

$\Longleftarrow$ The claim follows from Proposition 2.3.

$\Longrightarrow$ We proceed in several steps.

First we show that, for any $g(s) = \frac{p(s)}{q(s)}$ with $p(s) \in \mathbb{C}[s]$ and $q(s) \in \mathbb{C}[s]\backslash\{0\}$, we have

$$g(s) \overset{\text{ge}}{\approx} 1 \Longleftrightarrow \left[\forall\, z \in \mathbb{C} \,:\, p(z)p^\star(z) = q(z)q^\star(z)\right]. \quad (2.4)$$

This is a consequence of $\overline{p(\imath\omega)} = \overline{p}(-\imath\omega)$ and

$$\begin{aligned}\frac{p(s)}{q(s)} \overset{\text{ge}}{\approx} 1 &\overset{(1.2)}{\Longleftrightarrow} \forall\,\omega > 0 \;:\; p(\imath\omega)\overline{p}(-\imath\omega) = q(\imath\omega)\overline{q}(-\imath\omega)\\ &\overset{(2.2)}{\Longleftrightarrow} \forall\,\omega > 0 \;:\; p(\imath\omega)p^\star(\imath\omega) = q(\imath\omega)q^\star(\imath\omega)\\ &\Longleftrightarrow \forall\, z \in \mathbb{C} \;:\; p(z)p^\star(z) = q(z)q^\star(z)\,,\end{aligned}$$

where the last equivalence follows from the identity property of analytic functions, see for example [3].

Next we show that, for any $g(s) = \frac{p(s)}{q(s)}$ with coprime $p(s), q(s) \in \mathbb{C}[s]$ and $g(s) \overset{\text{ge}}{\approx} 1$ we have

$$p(z) = 0 \iff q(-\bar{z}) = 0 \qquad \forall z \in \mathbb{C}, \quad (2.5)$$

$$\{z \in \mathbb{C} \,|\, p(z) = p(-\bar{z}) = 0\} = \emptyset. \quad (2.6)$$

Note that $p(s), q(s) \in \mathbb{C}[s]$ are coprime (or, equivalently, solving the Bézout equation, there exist $\alpha(s), \beta(s) \in \mathbb{C}[s]$ such that $\alpha(s)p(s) + \beta(s)q(s) = 1$) if, and only if, $p^\star(s), q^\star(s)$ are coprime. Therefore, for any $z \in \mathbb{C}$,

$$p(z) = 0 \overset{(2.4)}{\Leftrightarrow} q^\star(z) = 0 \overset{(2.2)}{\Leftrightarrow} \overline{\overline{q}(-z)} = 0 \Leftrightarrow \quad q(-\bar{z}) = 0\,.$$

This proves (2.5). To show Assertion (2.6), assume that $p(z) = 0 = p(-\bar{z})$ for some $z \in \mathbb{C}$. Then Assertion (2.5) yields that $q(z) = 0$, and this contradicts coprimeness of $p(s)$, $q(s)$; hence Assertion (2.6) follows.

Finally, assume that $g(s)$ is factorized as

$$g(s) = r\,\mathrm{e}^{\imath\varphi}\,\frac{p(s)}{q(s)} \quad \text{for} \quad \begin{array}{l}\text{monic } p(s), q(s) \in \mathbb{C}[s],\\ \varphi \in [0, 2\pi),\ r > 0.\end{array}$$

Then $r = 1$ by $g(s) \overset{\text{ge}}{\approx} 1$, and (2.5) yields $q(s) = p^\star(s)$, and (2.6) gives $p(\imath\mathbb{R}) \cap \{0\} = \emptyset$.

This completes the proof of the proposition. $\square$

2.2 Factorization

Consider a stable transfer function

$$g(s) = \frac{p(s)}{q_{\mathrm{H}}(s)} \qquad \begin{array}{l}p(s), q_{\mathrm{H}}(s) \in \mathbb{C}[s] \text{ coprime},\\ p(\imath\mathbb{R}) \cap \{0\} = \emptyset\,,\end{array}$$

where for technical reasons we have assumed the the polynomials are coprime and the numerator polynomials does not have any zeros on the imaginary axis. Let $p(s)$ be uniquely factorized into

$$p(s) = \gamma\, p_{\mathrm{H}}(s)\, p_{\mathrm{aH}}(s), \text{ where } \begin{array}{l}\gamma \in \mathbb{C},\\ p_{\mathrm{H}}(s) \text{ is monic and Hurwitz},\\ p_{\mathrm{aH}}(s) \text{ is monic and anti-Hurwitz}.\end{array}$$

Then we may factorize

$$\frac{p(s)}{q_{\mathrm{H}}(s)} = \gamma\,\frac{p_{\mathrm{H}}(s)\; p^\star_{\mathrm{aH}}(s)}{q_{\mathrm{H}}(s)} \cdot \frac{p_{\mathrm{aH}}(s)}{p^\star_{\mathrm{aH}}(s)}\,. \quad (2.7)$$

Note that, in view Proposition 2.4, $p_{\mathrm{aH}}(s)/p^\star_{\mathrm{aH}}(s)$ is an all-pass and the numerator and denominator polynomials of the remaining factor $\gamma\, p_{\mathrm{H}}(s)\; p^\star_{\mathrm{aH}}(s)/q_{\mathrm{H}}(s)$ are both Hurwitz polynomials. Moreover, it is easy to see that the factorization in (2.7) is indeed unique in the following sense: Suppose

$$\frac{p(s)}{q_{\mathrm{H}}(s)} = \tilde{\gamma} \cdot \frac{\tilde{p}_{\mathrm{H}}(s)}{\tilde{q}_{\mathrm{H}}(s)} \cdot \frac{\pi(s)}{\pi^\star(s)},$$

where $\tilde{\gamma} \in \mathbb{C}$, $\tilde{p}_{\mathrm{H}}(s)$, $\tilde{q}_{\mathrm{H}}(s)$, $\pi^\star(s) \in \mathbb{C}[s]$ are monic and Hurwitz. Then

$$\underbrace{\gamma\; p_{\mathrm{aH}}(s)\;\left[\; p_{\mathrm{H}}(s)\right.}_{=p(s)}\; \left.\tilde{q}_{\mathrm{H}}(s)\, \pi^\star(s)\;\right] = \left[\;\tilde{\gamma}\, q_{\mathrm{H}}(s)\, \tilde{p}_{\mathrm{H}}(s)\;\right] \pi(s)$$

and since all polynomials are monic it follows that $\gamma = \tilde{\gamma}$; since the polynomials in parenthesis are Hurwitz and the remaining are anti-Hurwitz it follows that $\pi(s) = p_{\mathrm{aH}}(s)$ is uniquely determined; finally, coprimeness of $p(s)$ and $q(s)$ together with $\frac{\tilde{p}_{\mathrm{H}}(s)}{\tilde{q}_{\mathrm{H}}(s)} = \frac{p(s)\,\pi^\star(s)}{q_{\mathrm{H}}(s)\,\pi(s)}\gamma^{-1}$ shows that $\tilde{p}_{\mathrm{H}}(s)$ and $\tilde{q}_{\mathrm{H}}(s)$ are uniquely defined.

In the following theorem we will show that the above observation holds in a general context. We stress that the factorization is well-known; however, uniqueness is usually not explicitly mentioned.

Theorem 2.5 *For any weakly stable transfer function* $\frac{p(s)}{q(s)} \in \mathbb{C}(s)\backslash\{0\}$, *there exists a unique factorization*

$$\boxed{\frac{p(s)}{q(s)} = \gamma \cdot \frac{p_\imath(s)}{q_\imath(s)} \cdot \frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)} \cdot \frac{\pi(s)}{\pi^\star(s)}} \quad (2.8)$$

such that $\gamma \in \mathbb{C}$ *and*

$p_\imath(s), q_\imath(s), p_{\mathrm{H}}(s), q_{\mathrm{H}}(s), \pi(s) \in \mathbb{C}[s]\backslash\{0\}$ *are monic*

$p_\imath(s), q_\imath(s)$ *are coprime and have zeros on* $\imath\mathbb{R}$ *only,*

$p_{\mathrm{H}}(s), q_{\mathrm{H}}(s)$ *are coprime and Hurwitz*

$\pi(s)^*$ *is Hurwitz.*

The unique factorization in (2.8) *can be constructed as follows: Factorize* $p(s)$ *and* $q(s)$ *uniquely into*

$$p(s) = \gamma_p\, p_\imath(s)\, p_{\mathrm{s}}(s)\, p_{\mathrm{u}}(s), \quad q(s) = \gamma_q\, q_\imath(s)\, q_{\mathrm{s}}(s)\,,$$

where $\gamma_p, \gamma_q \in \mathbb{C}$ *and*

$p_\imath(s), q_\imath(s), p_{\mathrm{s}}(s), q_{\mathrm{s}}(s), p_{\mathrm{u}}(s) \in \mathbb{C}[s]\backslash\{0\}$ *are monic*

$p_\imath(s), q_\imath(s)$ *have zeros on* $\imath\mathbb{R}$ *only,*

$p_{\mathrm{s}}(s), q_{\mathrm{s}}(s), p^*_{\mathrm{u}}(s)$ *are Hurwitz*

Then the factorization in (2.8) *is given by*

$$\begin{aligned}p_{\mathrm{H}}(s) &:= \frac{p_{\mathrm{s}}(s)p^\star_{\mathrm{u}}(s)}{\gcd\big(p_{\mathrm{s}}(s)p^\star_{\mathrm{u}}(s), q_{\mathrm{s}}(s)\big)},\\ q_{\mathrm{H}}(s) &:= \frac{q_{\mathrm{s}}(s)}{\gcd\big(p_{\mathrm{s}}(s)p^\star_{\mathrm{u}}(s), q_{\mathrm{s}}(s)\big)}\,,\\ \pi(s) &:= p_{\mathrm{u}}(s)\,, \qquad \gamma := \gamma_p/\gamma_q\,,\end{aligned}$$

where $\gcd\big(r(s), \hat{r}(s)\big) \in \mathbb{C}[s]$ *denotes the monic greatest common divisor of* $r(s), \hat{r}(s) \in \mathbb{C}[s]$.

Proof Existence of a factorization (2.8) with the required properties for all polynomials follows from the definitions in the statement of the theorem.

We show uniqueness of the factorization (2.8): Substituting the factorized polynomials $p(s)$ and $q(s)$ into (2.8), we obtain

$$\begin{aligned}\gamma_p\, p_\imath(s)\, p_{\mathrm{s}}(s)\, p_{\mathrm{u}}(s) \,\cdot\, q_{\mathrm{H}}(s)\, \pi^\star(s) \\ = \gamma_q\, q_\imath(s)\, q_{\mathrm{s}}(s) \,\cdot\, \gamma\, p_\imath(s)\, p_{\mathrm{H}}(s)\, \pi(s)\,,\end{aligned}$$

and since all polynomials are monic, we conclude $\gamma = \gamma_p/\gamma_q$. Now cancelation gives

$$p_{\mathrm{u}}(s)\left[p_{\mathrm{s}}(s)\, q_{\mathrm{H}}(s)\, \pi^\star(s)\right] \;=\; \left[q_{\mathrm{s}}(s)\, p_{\mathrm{H}}(s)\right] \pi(s)\,.$$

Since the polynomials in parenthesis are Hurwitz and $p_{\mathrm{u}}(s)$, $\pi(s)$ are anti-Hurwitz, we conclude that $\pi(s) = p_{\mathrm{u}}(s)$ is unique and thus

$$p_{\mathrm{s}}(s)\, q_{\mathrm{H}}(s)\, \pi^\star(s) \;=\; q(s)\, p_{\mathrm{H}}(s)\,.$$

This gives

$$\frac{\dfrac{p_{\mathrm{s}}(s) p_{\mathrm{u}}^\star(s)}{\gcd\left(p_{\mathrm{s}}(s) p_{\mathrm{u}}^\star(s), q_{\mathrm{s}}(s)\right)}}{\dfrac{q_{\mathrm{s}}(s)}{\gcd\left(p_{\mathrm{s}}(s) p_{\mathrm{u}}^\star(s), q_{\mathrm{s}}(s)\right)}} \;=\; \frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)}\,,$$

and since the quotients on the left hand side are coprime, uniqueness of $p_{\mathrm{H}}(s)$ and $q_{\mathrm{H}}(s)$ follows. This completes the proof of the theorem. □

We stress that in view of Proposition 2.4, the right most factor in (2.8) satisfies $\frac{\pi(s)}{\pi^\star(s)} \in [1]_{\mathrm{ge}}$; therefore, this factor only changes the phase of $\omega \mapsto \frac{p(\imath\omega)}{q(\imath\omega)}$ and not the gain.

Theorem 2.5 allows to parameterize each equivalence class $\left[\frac{p(s)}{q(s)}\right]_{\mathrm{ge}}$ of a weakly stable transfer function $\frac{p(s)}{q(s)} \in \mathbb{C}(s)\backslash\{0\}$ by a number of modulus 1 and an all-pass as follows.

Proposition 2.6 *Let* $\frac{p(s)}{q(s)} \in \mathbb{C}(s)\backslash\{0\}$ *be a weakly stable transfer function factorized as in* (2.8)*. Then* $\tilde g(s)$ *is a gain equivalent weakly stable transfer function if, and only if, it has the form*

$$\tilde g(s) = \mathrm{e}^{\imath\varphi}\gamma \cdot \frac{p_\imath(s)}{q_\imath(s)} \cdot \frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)} \cdot \frac{\tilde\pi(s)}{\tilde\pi^\star(s)} \tag{2.9}$$

for some $\varphi \in [0, 2\pi)$ *and some monic and Hurwitz* $\tilde\pi^*(s) \in \mathbb{C}[s]\backslash\{0\}$.

Proof If $\tilde g(s)$ has the form (2.9), then it is weakly stable and gain equivalent by Proposition 2.4.

Conversely, consider a gain equivalent weakly stable transfer function. In view of Theorem 2.5, we may assume that

$$\frac{p(s)}{q(s)} \overset{\mathrm{ge}}{\sim} \tilde\gamma \cdot \frac{\tilde p_\imath(s)}{\tilde q_\imath(s)} \cdot \frac{\tilde p_{\mathrm{H}}(s)}{\tilde q_{\mathrm{H}}(s)} \cdot \frac{\tilde\pi(s)}{\tilde\pi^\star(s)}$$

where $\tilde\gamma \in \mathbb{C}$ and
$\tilde p_\imath(s), \tilde q_\imath(s), \tilde p_{\mathrm{H}}(s), \tilde q_{\mathrm{H}}(s), \tilde\pi(s) \in \mathbb{C}[s]\backslash\{0\}$ are monic
$\tilde p_{\mathrm{H}}(s), \tilde q_{\mathrm{H}}(s)$ are coprime and Hurwitz
$\tilde p_\imath(s), \tilde q_\imath(s)$ are coprime and have zeros on $\imath\mathbb{R}$ only,
$\tilde\pi(s)^*$ is Hurwitz.

Then gain equivalence together with Proposition 2.3 yields

$$\begin{aligned}&\left|\gamma\, \frac{p_\imath(\imath\omega)}{q_\imath(\imath\omega)}\, \frac{p_{\mathrm{H}}(\imath\omega)}{q_{\mathrm{H}}(\imath\omega)}\right| \\ &\quad = \left|\tilde\gamma\, \frac{\tilde p_\imath(\imath\omega)}{\tilde q_\imath(\imath\omega)}\, \frac{\tilde p_{\mathrm{H}}(\imath\omega)}{\tilde q_{\mathrm{H}}(\imath\omega)}\right| \quad \text{for almost all } \omega > 0.\end{aligned} \tag{2.10}$$

Since all polynomials in (2.10) are monic, we conclude that $|\gamma| = |\tilde\gamma|$ and hence there exists $\varphi \in [0, 2\pi)$ such that $\tilde\gamma = \mathrm{e}^{\imath\varphi}\gamma$. Now the identity property of analytic functions, see for example [3], gives

$$\begin{aligned}&\left|\frac{p_\imath(s)}{q_\imath(s)}\, \frac{p_{\mathrm{H}}(s)}{q_{\mathrm{H}}(s)}\right| \\ &\quad = \left|\frac{\tilde p_\imath(s)}{\tilde q_\imath(s)}\, \frac{\tilde p_{\mathrm{H}}(s)}{\tilde q_{\mathrm{H}}(s)}\right| \quad \text{for almost all } s \in \mathbb{C}.\end{aligned} \tag{2.11}$$

Since $p_\imath(s), q_\imath(s) \tilde p_\imath(s), \tilde q_\imath(s)$ have zeros on $\imath\mathbb{R}$ only, and $p_{\mathrm{H}}(s), q_{\mathrm{H}}(s), \tilde p_{\mathrm{H}}(s), \tilde q_{\mathrm{H}}(s)$ are Hurwitz, and the fractions are all coprime, it follows from (2.11) that

$$p_\imath(s) = \tilde p_\imath(s), q_\imath(s) = \tilde q_\imath(s), p_{\mathrm{H}}(s) = \tilde p_{\mathrm{H}}(s), q_{\mathrm{H}}(s) = \tilde q_{\mathrm{H}}(s).$$

This completes the proof of the proposition. □

2.3 Main result

We will now state the main result.

Theorem 2.7 *For any weakly stable transfer function* $\frac{p(s)}{q(s)} \in \mathbb{C}(s)\backslash\{0\}$ *we have:*

$$\begin{aligned}\frac{p(s)}{q(s)} \text{ is min. phase} \quad &\Longleftrightarrow \quad p(s) \text{ is weakly Hurwitz} \\ &\Longleftrightarrow \quad \pi(s) \equiv 1 \quad \text{in } (2.8)\,.\end{aligned}$$

Proof Let $\frac{p(s)}{q(s)}$ be factorized as in (2.8). Denote the roots of $p_\imath(\imath\cdot)$ and $q_\imath(\imath\cdot)$ on $\imath\mathbb{R}$ by

$$\mathcal{R}(p_\imath q_\imath) \;:=\; \{\omega \in \mathbb{R}\,|\, p_\imath(\imath\omega) q_\imath(\imath\omega) = 0\}$$

and choose the argument functions

$$\begin{aligned}\arg(p/q)(\imath\cdot) &: \mathbb{R}\backslash\mathcal{R}(p_\imath q_\imath) \to \mathbb{R} \\ \arg p_\imath(\imath\cdot) &: \mathbb{R}\backslash\mathcal{R}(p_\imath q_\imath) \to \mathbb{R} \\ \arg(p_{\mathrm{H}}/q_{\mathrm{H}})(\imath\cdot) &: \mathbb{R} \to \mathbb{R} \\ \arg \pi(\imath\cdot) &: \mathbb{R} \to \mathbb{R} \\ \arg \pi^\star(\imath\cdot) &: \mathbb{R} \to \mathbb{R}\,.\end{aligned}$$

We proceed in several steps.
Step 1: If $s - s_0$ is Hurwitz, then an elementary geometric argument yields that $\frac{\mathrm{d}}{\mathrm{d}\omega}\arg(\imath\omega - s_0) > 0$ for all $\omega \in \mathbb{R}$. This is the "phase increasing property" which

also holds for any Hurwitz polynomial, see [4, Proposition 3.4.5]. Therefore, if $\deg \pi(s) > 0$, then the Hurwitz polynomial $\pi^\star(s)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \arg \pi^\star(\imath\omega) \; > \; 0 \qquad \forall\, \omega \in \mathbb{R}\,,$$

and since $\pi(s)$ is anti-Hurwitz, we may derive similarly that

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \arg \pi(\imath\omega) \; < \; 0 \qquad \forall\, \omega \in \mathbb{R}\,.$$

Therefore we arrive, if $\deg \pi(s) > 0$, at

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \arg \frac{\pi(\imath\omega)}{\pi^*(\imath\omega)} < 0 \qquad \forall\, \omega \in \mathbb{R}\,. \tag{2.12}$$

Step 2: Note that (1.3) yields

$$\frac{\mathrm{d}}{\mathrm{d}\omega} \arg \frac{p_\imath(\imath\omega)}{q_\imath(\imath\omega)} = 0 \qquad \forall\, \omega \in \mathbb{R}\backslash\mathcal{R}(p_\imath q_\imath)\,. \tag{2.13}$$

Step 3: Now we conclude, for all $\omega \in \mathbb{R}\backslash\mathcal{R}(p_\imath q_\imath)$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\omega} \arg \frac{p(\imath\omega)}{q(\imath\omega)} \\
&\quad\overset{(2.8)}{=} \frac{\mathrm{d}}{\mathrm{d}\omega} \arg \left(\gamma \, \frac{p_\imath(\imath\omega)}{q_\imath(\imath\omega)} \, \frac{p_{\mathrm{H}}(\imath\omega)}{q_{\mathrm{H}}(\imath\omega)} \, \frac{\pi(\imath\omega)}{\pi^\star(\imath\omega)} \right) \\
&\quad\overset{(2.13)}{=} \frac{\mathrm{d}}{\mathrm{d}\omega} \arg \left(\gamma \, \frac{p_{\mathrm{H}}(\imath\omega)}{q_{\mathrm{H}}(\imath\omega)} \right) + \frac{\mathrm{d}}{\mathrm{d}\omega} \arg \left(\frac{\pi(\imath\omega)}{\pi^*(\imath\omega)} \right)
\end{aligned} \tag{2.14}$$

Step 4: We are now ready to finalize the proof. The equivalence of $\frac{p(s)}{q(s)}$ being minimum phase and $\pi(s) \equiv 1$ is a consequence of Proposition 2.6 together with (2.12) and (2.14).
The equivalence of $p(s)$ having zeroes in the closed left half complex plane only and $\pi(s) \equiv 1$ is a consequence of the definition of $\pi(s)$ in Theorem 2.5. This completes the proof of the theorem. $\square$

Theorem 2.7 allows to show that the factorization (2.8) is closely related to the so called inner-outer factorization, see [2, Sect. 2.8]. However, their setup is more restrictive since only stable and proper transfer functions are allowed. To explain this concept in more detail, we need to generalize the Hurwitz restriction to rational functions as follows:

$$g^*(s) = \frac{p^*(s)}{q^*(s)} \qquad \text{for any } g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)\backslash\{0\}\,.$$

Definition 2.8 *Let $g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)\backslash\{0\}$ be a stable and proper transfer function, i.e. $q(s)$ is Hurwitz and* $\deg p(s) \le \deg q(s)$. *Then $g(s)$ is called*

$$\begin{aligned}
&\text{inner} \;:\Longleftrightarrow\; g^*(s)\, g(s) \equiv 1 \\
&\text{outer} \;:\Longleftrightarrow\; \exists\, x(s) \in \mathbb{C}(s) \textit{ analytic in } \mathbb{C}_+ \\
&\qquad\qquad\quad \textit{such that } \; g(s)\, x(s) = 1\,.
\end{aligned}$$

Note that in [2] the notation $g\tilde{}(s)$ is used instead of $g^*(s)$.
The notion of inner is the same as a stable all-pass:

Proposition 2.9 *Any stable and proper $g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)\backslash\{0\}$ satisfies:*

(i) $\quad g(s)$ is inner $\iff$ $g(s)$ is an all-pass.
(ii) $\quad g(s)$ is outer $\iff$ $g(s)$ is minimum phase.

Proof (i) Write

$$g(s) = \gamma \frac{\hat{p}(s)}{\hat{q}(s)} \quad \text{for } \gamma \in \mathbb{C} \text{ and monic } \hat{p}(s),\, \hat{q}(s) \in \mathbb{C}[s].$$

If $g(s)$ is inner, then

$$\left| \gamma \, \frac{\hat{p}(s)}{\hat{q}(s)} \;\cdot\; \bar{\gamma} \, \frac{\hat{p}^*(s)}{\hat{q}^*(s)} \right| = 1 \qquad \text{for almost all } s \in \mathbb{C}$$

and since $\hat{p}(s)$, $\hat{q}(s)$ are monic, it follows that $|\gamma\bar{\gamma}| = 1$ and so $|\gamma| = 1$. Therefore, $|p(s)/q(s)| = 1$ for almost all $s \in \mathbb{C}$ and thus $g(s)$ is an all-pass.
If $g(s)$ is an all-pass, then Proposition 2.3 yields that $g^*(s)$ is an all-pass; this gives $|g(\imath\omega)\, g^*(\imath\omega)| = 1$ for almost all $\omega > 0$. Now the identity property of analytic functions yields that $g(s)$ is inner.
(ii) This equivalence is immediate from the definition of outer and Theorem 2.7. $\square$

3 Zero dynamics

The concept of asymptotically stable zero dynamics is closely related to minimum phase. We will show that if a minimal realization of a proper transfer function $\frac{p(s)}{q(s)} \in \mathbb{K}(s) \setminus \{0\}$ has asymptotically stable zero dynamics, then $\frac{p(s)}{q(s)}$ is minimum phase. However, the converse implication does, in general, not hold true.
The zero dynamics and its stability is defined for systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + bu(t), \\ y(t) &= cx(t), \end{aligned} \tag{3.1}$$

where $(A, b, c) \in \mathbb{K}^{n\times n} \times \mathbb{K}^n \times \mathbb{K}^{1\times n}$, the space of trajectories is a subset of $\mathcal{X} := \mathcal{AC}(\mathbb{R}_{\ge 0}; \mathbb{R}^n)$, i.e. the set of absolutely continuous functions $x : \mathbb{R}_{\ge 0} \to \mathbb{R}^n$, see [4, Def. A.3.12]; and the input space is $\mathcal{U} := \mathcal{PC}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$, i.e. the set of piecewise continuous functions $u : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$, that means u is left continuous and has only finitely many discontinuities on any compact subset of $\mathbb{R}_{\ge 0}$.

Definition 3.1 *The* zero dynamics *of system* (3.1) *is defined as the set of trajectories*

$$\mathcal{ZD} \;:=\; \left\{ (x, u) \in \mathcal{X} \times \mathcal{U} \;\middle|\; \begin{array}{l} (x, u) \textit{ solves } (3.1) \\ \textit{on } \mathbb{R}_{\ge 0} \textit{ a.e.} \\ \textit{such that } y \equiv 0 \end{array} \right\}.$$

The zero dynamics are called asymptotically stable *if, and only if,*

$$\forall\,(x,u) \in \mathcal{ZD} \;:\; \lim_{t\to\infty} (x(t),u(t)) \;=0\,.$$

By linearity of (3.1), the set $\mathcal{ZD}$ is a real vector space and it can be shown that it carries the structure of a linear dynamical system as, for example, defined in [4, Definition 2.1.1].

It also can be shown that the zero trajectory $(x,u)=0$ of $\mathcal{ZD}$ is attractive and stable if, and only if, $\mathcal{ZD}$ is asymptotically stable.

Theorem 3.2 *Let $(A,b,c) \in \mathbb{K}^{n\times n}\times\mathbb{K}^n\times\mathbb{K}^{1\times n}$ be a minimal realization (i.e. controllable and observable) of a weakly stable transfer function $\frac{p(s)}{q(s)} \in \mathbb{C}(s)\backslash\{0\}$.*
Then (3.1) has asymptotically stable zero dynamics if, and only if, $p(s)$ is Hurwitz.

An immediate consequence of Theorem 3.2 is that asymptotically stable zero dynamics yields that $\frac{p(s)}{q(s)}$ is minimum phase; and furthermore, that the converse implication is, in general, not true.

Before we prove Theorem 3.2, an essential decomposition of a transfer function and its realization will be treated in the next proposition. In fact, this is a canonical form as we will show in (4.3).

The decomposition is depicted in Figure 1.3.1; it separates $y(s)=\gamma\,\frac{\tilde p(s)}{\tilde q(s)}\,u(s)$ into a ρ-times integrator in combination with an "internal loop" $y(s)\mapsto \hat y(s) = -\frac{\beta(s)}{\tilde p(s)}y(s)$. For various control design purposes, it is important that the zeros of $\tilde p(s)$ determine completely the internal stability properties of the system.

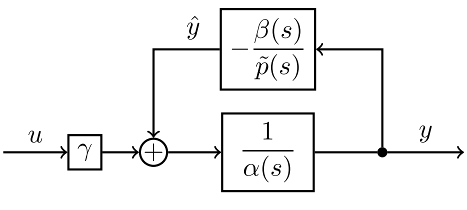

Bild 1.3.1: Decomposition (4.2) of the system (4.1)

Proposition 3.3 *Any proper transfer function $\frac{p(s)}{q(s)} \in \mathbb{K}(s)\backslash\{0\}$ can uniquely be written in the form*

$$\frac{p(s)}{q(s)} \;=\; \gamma\,\frac{\tilde p(s)}{\alpha(s)\;\tilde p(s)+\beta(s)}\,, \tag{3.2}$$

where

$$\left.\begin{array}{ll}
\gamma = \lim\limits_{s\to\infty} \frac{s^\rho p(s)}{q(s)}\,, & \rho = \deg q(s) - \deg p(s)\\
\tilde p(s) \in \mathbb{K}[s]_{\text{monic}} \quad \textit{with} & \deg\tilde p(s) = n-\rho\\
\beta(s) \in \mathbb{K}[s]_{\text{monic}} \quad \textit{with} & \deg\beta(s) < \deg\tilde p(s)\\
\alpha(s) = s^\rho - \sum\limits_{i=1}^{\rho} a_i s^{i-1} \in \mathbb{K}[s] & \\
\tilde p(s),\ \alpha(s)\tilde p(s)+\beta(s) \textit{ are coprime}\,. &
\end{array}\right\} \tag{3.3}$$

Moreover, a minimal realization of $\frac{p(s)}{q(s)}$ is given by

$$A=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & & & 0\\ \vdots & & \ddots & \ddots & & \vdots\\ 0 & 0 & \cdots & 0 & 1 & 0\\ a_1 & a_2 & \cdots & a_{\rho-1} & a_\rho & S\\ P & 0 & \cdots & 0 & 0 & Q\end{bmatrix}, b=\begin{bmatrix}0\\0\\ \vdots\\0\\ \gamma\\ 0_{n-\rho}\end{bmatrix}, c=\begin{bmatrix}1\\0\\ \vdots\\0\\0\end{bmatrix}^\top \tag{3.4}$$

for some $a_1,\dots,a_\rho\in\mathbb{K}$, $\gamma\in\mathbb{K}\backslash\{0\}$, $Q\in\mathbb{K}^{(n-\rho)\times(n-\rho)}$, $P, S^\top \in \mathbb{K}^{(n-\rho)\times 1}$, such that

$$\frac{\beta(s)}{\tilde p(s)} \;=\; S\,(sI_{n-\rho}-Q)^{-1}\,P\,.$$

In addition, the realization (Q,P,S) is a minimal.

Proof Choose unique coprime and monic $\tilde p(s), \tilde q(s) \in \mathbb{K}[s]$ such that

$$g(s) = \gamma\,\frac{\tilde p(s)}{\tilde q(s)}\,. \tag{3.5}$$

Then long division yields

$$\tilde q(s) = \alpha(s)\,\tilde p(s)+\beta(s) \;\text{ for some } \alpha(s),\beta(s)\in\mathbb{K}[s]$$
$$\text{so that }\; \deg\beta(s) < \deg\tilde p(s).$$

Therefore, (3.2) holds; and it is easy to see that polynomials and numbers in (3.2) are unique and the properties in (3.3) hold.

Let

$$\begin{aligned}\dot\eta &= Q\,\eta + P\,y\\ \hat y &= S\,\eta\end{aligned}$$

be a minimal realization of $\hat y(s) = \frac{\beta(s)}{\tilde p(s)}y(s)$ which has state space dimension $\deg\tilde p(s) = n-\rho$. Then with the choice

$$x(t) := \Big(y(t), y^{(1)}(t),\dots,y^{(\rho-1)}(t),\eta(t)^\top\Big)^\top\,,$$

a realization of (3.2) is given by (3.4).

It remains to show that (3.4) is controllable and observable. Controllability is a consequence of

$$[b, Ab,\dots,A^{n-1}] = \left[\begin{array}{c|ccc} & k_{1,\rho+1} & \cdots & k_{1,n}\\ \gamma I_\rho & \vdots & & \vdots\\ & k_{\rho,\rho+1} & \cdots & k_{\rho,n}\\ \hline 0_{(n-\rho)\times\rho} & \gamma P, & \dots, & K(P,Q)\end{array}\right]$$

where

$$K(P,Q) = \gamma P, \ldots, k_{1,n-1}P + \ldots + k_{1,n-\rho-2}Q^{n-\rho-2}P + \gamma Q^{n-\rho-1}P$$

and the matrix has full rank since (Q,P) is controllable. Similarly, observability follows from

$$\begin{bmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{bmatrix} = \left[\begin{array}{ccc|c} & I_\rho & & 0_{\rho\times(n-\rho)} \\ \hline x_{\rho+1,1} & \ldots & x_{\rho+1,\rho} & S \\ x_{\rho+2,\rho} & \ldots & x_{\rho+2,\rho} & x_{\rho+1,\rho}S + SQ \\ \vdots & & \vdots & \vdots \\ x_{n,\rho} & \ldots & x_{n,\rho} & X(S,Q) \end{array}\right]$$

where

$$X(S,Q) = x_{n-1,\rho}S + \ldots + x_{n-\rho-1,\rho}SQ^{n-\rho-2} + SQ^{n-\rho-1},$$

and the matrix has full rank since (S,Q) is observable. This completes the proof of the proposition. $\square$

Although of no relevance here, we note that the system (3.4) is in Bynres-Isidori form; see [6, Sec. 4.1].

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2: Asymptotic stability of the zero dynamics is invariant under state space transformation; this follows immediately from the definition. Therefore, without restriction of generality we may assume that the system (3.1) is in the form (3.4). The zero dynamics of (3.4) are given by

$$\mathcal{ZD} = \left\{ \left(\left(\begin{smallmatrix} 0_\rho \\ \eta(\cdot) \end{smallmatrix}\right),\ -(cA^{\rho-1}b)^{-1}S\eta(\cdot) \right) \,\middle|\, \dot\eta = Q\eta \right\},$$

Since (Q,P,S) is a minimal realization of $\frac{\beta(s)}{\tilde p(s)}$, it follows that

$$\{ s\in\mathbb{C} \,|\, p(s) = 0 \} \;=\; \sigma(Q)\,.$$

This completes the proof of the theorem. $\square$

4 Canonical forms

The general notion of canonical form studies simple representatives of equivalence classes in a given set.

Definition 4.1 *Let $\mathcal{M}$ be a nonempty set and $\sim$ an equivalence relation on $\mathcal{M}\times\mathcal{M}$. A map $\Gamma:\mathcal{M}\to\mathcal{M}$ is called a* canonical form for the equivalence relation $\sim$ *if, and only if,*

$$\forall\, m,m'\in\mathcal{M} \,:\, \Gamma(m)\sim m \;\wedge\; [m\sim m' \Leftrightarrow \Gamma(m) = \Gamma(m')]\,.$$

This means that the set $\mathcal{M}$ is divided into the disjoint equivalence classes and the mapping Γ picks a unique representative in each class.

We first consider a canonical form for gain equivalent weakly stable transfer functions. denoted by

$$\mathbb{C}(s)_{\rm ws} \;:=\; \left\{ \tfrac{p(s)}{q(s)} \in \mathbb{C}(s)\backslash\{0\} \,\middle|\, q(s) \text{ is weakly Hurwitz} \right\}.$$

Proposition 4.2 *The map*

$$\begin{array}{rccc} \Gamma: & \mathbb{C}(s)_{\rm ws} & \to & \mathbb{C}(s)_{\rm ws} \\ & \gamma\,\dfrac{p_\imath(s)}{q_\imath(s)}\,\dfrac{p_{\rm H}(s)}{q_{\rm H}(s)}\,\dfrac{\pi(s)}{\pi^\star(s)} & \mapsto & |\gamma|\,\dfrac{p_\imath(s)}{q_\imath(s)}\,\dfrac{p_{\rm H}(s)}{q_{\rm H}(s)}\,, \end{array}$$

where we use the unique factorization (2.8)*, is a canonical form for $\overset{\rm ge}{\approx}$ on $\mathbb{C}(s)_{\rm ws}$.*

Proof It follow from the definition of Γ and Proposition 2.4 that $g(s) \overset{\rm ge}{\approx} \Gamma(g(s))$. Furthermore, by Proposition 2.6 we have for any $\tilde g(s)\in\mathbb{C}(s)_{\rm ws}$ that

$$\gamma\,\frac{p_\imath(s)}{q_\imath(s)}\,\frac{p_{\rm H}(s)}{q_{\rm H}(s)}\,\frac{\pi(s)}{\pi^\star(s)} \quad \overset{\rm ge}{\approx} \quad \tilde g(s)$$

if, and only if,

$$\tilde g(s) = {\rm e}^{\imath\varphi}\gamma\cdot\frac{p_\imath(s)}{q_\imath(s)}\cdot\frac{p_{\rm H}(s)}{q_{\rm H}(s)}\cdot\frac{\tilde\pi(s)}{\tilde\pi^\star(s)}$$

with $\tilde\pi^\star(s)$ Hurwitz. This is the case if, and only if, $\Gamma(g(s)) = \Gamma(\tilde g(s))$ and the proof is complete. $\square$

In prose, Proposition 4.2 may be described as follows: Within each equivalence class $[g(s)]_{\rm ge}$ of a weakly stable transfer function $g(s) = \frac{p(s)}{q(s)} \in \mathbb{C}(s)$ there exists a unique transfer function which is minimum phase. Or, in other words, the minimum phase system is the canonical representative within its equivalence class. In view of the factorization (2.8), a possible canonical representative is $\widehat g(s) := |\gamma|\cdot\frac{p_\imath(s)}{q_\imath(s)}\cdot\frac{p_{\rm H}(s)}{q_{\rm H}(s)}$, but could equally be chosen as ${\rm e}^{\imath\varphi}\gamma\cdot\frac{p_\imath(s)}{q_\imath(s)}\cdot\frac{p_{\rm H}(s)}{q_{\rm H}(s)}$ for any $\varphi\in[0,2\pi)$.

Finally, we turn to the decomposition (3.2) of a proper transfer function which is the key to prove Theorem 3.2. Consider again a system described by

$$y(s) = g(s)\; u(s) \qquad \text{for proper transfer function} \quad g(s) = \tfrac{p(s)}{q(s)} \in \mathbb{K}(s)\backslash\{0\}\,. \tag{4.1}$$

Then invoking the decomposition of Proposition 3.3, a little algebraic manipulation allows to rewrite the input-output system (4.1) as

$$y(s) == \tfrac{1}{\alpha(s)} \Big[\underbrace{-\frac{\beta(s)}{\tilde p(s)}\, y(s) + \gamma\, u(s)}_{=:\hat y(s)} \Big]. \tag{4.2}$$

Note that $\deg\alpha(s) = \rho$.

It is now straightforward to check that for the equivalence class of equal rational functions, the mapping

$$\Gamma : \frac{p(s)}{q(s)} \mapsto \gamma \frac{\tilde{p}(s)}{\alpha(s)\ \tilde{p}(s) + \beta(s)} \tag{4.3}$$

as defined in (3.2) and depicted in Figure 1.3.1, is a canonical form on the set of proper rational functions.

Conclusions We have defined and characterized minimum phase of weakly stable transfer functions. It has been shown that asymptotical stable zero dynamics of a minimal realization of a transfer functions implies minimum phase, but not vice versa. It has also been shown that minimum phase systems is in canonical form within all gain equivalent transfer functions.

Danksagung

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